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NOTES ON CLASSICAL ANALOGS OF QUANTUM BLACK HOLES

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Abstract

The model is built in which the main global properties of classical and quasi-classical black holes become local (the event horizon, "no-hair," temperature and entropy). Our construction is based on the features of a quantum collapse, discovered when studying some quantum black hole models. But our model is purely classical, and this allows to use self-consistently the Einstein equations and classical (local) thermodynamics and explain in this way the "log 3"-puzzle.

Key words: classical and quasi-classical black holes.

Introduction

More than 80 years ago the famous soviet poet Vladimir Mayakovsky advised everybody (even the elderly black people) to study Russian only because Vladimir Lenin spoke this language (Vladimir Mayakovsky "To Our Youth," 1927). I met many old black men in my life but nobody can speak Russian, may be they never heard about Mayakovsky, and this very verse was not translated into English. The advice remained just a dream.

But, I indeed met one (now) elderly woman Lucette Carter, the wife to the famous relativist Brandon Carter, who was studying Russian when being a PhD student, only because she became aware of the book "Einstein Spaces" by Alexey Zinovievich Petrov published at that time only in Russian (A.Z. Petrov, "Prostranstva Einshteina," M.: Fizmatgiz, 1961), and decided to read it. And this is a reality.

About 65 years ago (1946), Alexey Zinovievich Petrov started his seminal investigations of the algebraic structure of the space-times – solutions to the vacuum Einstein equations. The result of the enormous efforts lasted for at least 15 years is known at present as the Petrov Classification of Gravitational Fields, Petrov types I, D, II, N and III in modern notations.

Of these, we are interested here in the degenerate Petrov type D. This is because all the black hole solutions belong to it. The appearance of black holes is a striking phenomenon, the origin of which lies in the relativistic character of the space-time, i.e., in the fundamental role played by the speed of light defining the causal structure, and in the equivalence between mass and energy. The latter feature tells us that in any self-consistent relativistic theory of gravity all the energy should gravitate, including the gravitational energy itself. In the most concentrated form these two effects reveal themselves in black holes. The black hole space-times have a rather unusual (from the point of view of our experience, or common sense) geometric and causal structure. Their physical properties are also impressive and, in fact, marginal. In the next section some of them will be briefly described. But now we would like to emphasize that all the unusual features of the black holes are that of the space-times themselves. Moreover, the quantized matter fields acquire, in the presence of black holes, some unexpected properties. This can be considered as the first step to the semi-classical quantization of the black holes space-times, and any future quantum theory of gravity, or “of everything,” should reproduce all these results. It is in this sense that black holes become a bridge between the classical General Relativity (or any other relativistic gravitational theory) and the overall quantum realm.

1. Preliminaries

Classical definition of a black hole is based on the existence of the event horizon [1] – the boundary of a space-time region from which the light cannot escape to infinity. The very notion of the event horizon is global and requires the knowledge of the whole history, both past and future.

Classical "black hole has no hair" [2] and is described by only few parameters: mass, Coulomb-like charge and angular momentum. The Schwarzschild black hole has only mass, the Reissner-Nordstrom one has mass and charge, the Kerr black hole has mass and angular momentum. The most general type – Kerr – Newman black hole - has all three parameters. This resembles the body in thermal equilibrium. The process of becoming bold is also global; its duration, formally, is infinite, like the process of coming to thermal equilibrium. It goes through radiating of all possible perturbations and governed by Schroedinger-like wave equation, first derived in [3]. The results of many numerical studies for a long period (two decades) were summarized in [4]. It appeared that such perturbation modes have discrete spectra with complex frequencies w. They received the name “quasi-normal frequencies.” The imaginary parts are equidistant indicating that the decaying modes are radiating away in a manner reminiscent of the last pure dying tones of a ringing bell, and the higher the overtone, the shorter its lifetime. The real part of quasi-normal frequencies tends to some constant value which depends on the black hole type. For Schwarzschild black holes we are interested in here $Gm\,w_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + O[(n+1)^{-1/2}]$, $n \to \infty$, where m is the mass, and G is the Newton’s constant. All that shows that black holes have some inherent frequency. Therefore, they are not “dead” but have some “private life,” encoded in some features of their horizons. Evidently, this is also a global property because it does not depend on what is going on inside.

Investigation of the processes near an event horizon showed that they can be reversible and irreversible like in thermodynamics [5, 6]. The assimilation of a point (classical) particle by a non-extremal (if a black hole has more than one parameter, then, for a fixed value of parameters other than mass, there exists the minimal value of mass – critical, or extreme – below which the event horizon does not exist) black hole reversible if it is injected at the event horizon from a radial turning point of its motion. In this case, the black hole (horizon) area remains unchanged, and the change in other parameters (mass, charge, and angular momentum) can be undone by another suitable (reversible) process. In all other cases, the horizon area A increases. Thus, for classical black holes $dA \geq 0$.

The new area in black hole physics started with the seminal paper by J.D. Bekenstein [7–9], where he presented serious physical arguments that the Schwarzschild black hole should be described by a certain amount of entropy which is proportional to the area of event horizon. Such a strict proportionality could appear to be playing games with symbols with only one parameter, black hole mass, but it was then confirmed by J.M. Bardeen, B. Carter and S.W. Hawking [10], who proved the four laws of thermodynamics for the general class of Kerr – Newman black holes. Moreover, it was shown that the role of temperature is played by the surface gravity $\varkappa$ at the event horizon (up to some numerical factor), which is constant there. And only after discovering by S.W. Hawking the black hole evaporation [11, 12], this thermodynamical analogy became the real physical phenomenon. He considered the quantum theory of massless scalar field on the Schwarzschild static space-time background and found that the specific boundary conditions (only infalling waves in the vicinity of the horizon) result in a thermal behavior of the wave functions and nonvanishing energy flow to the infinity. It appeared that the spectrum of such a radiation is Planckian with the temperature

$$T_H = \frac{\varkappa_H}{2\pi}, \tag{1}$$

where $\varkappa$ is the surface gravity at the event horizon. It follows, then, that the black hole entropy is exactly one-fourth of dimensionless horizon area

$$S = \frac{1}{4}\frac{A}{\ell_{p_\ell}^2}, \tag{2}$$

where $\ell_{p_\ell} = \sqrt{\dfrac{\hbar G}{c^3}} \sim 10^{-33}$ cm is the Planck length ($\hbar$ is the Planck constant, c is the speed of light, and G is the Newton's gravitational constant). We will use the units $\hbar = c = 1$, so $\ell_{p_\ell} = \sqrt{G}$ and the Planck mass is $m_{p_\ell} = \sqrt{\dfrac{\hbar c}{G}} = 1/\sqrt{G} \sim 10^{-5}$ g.

The nature of Hawking radiation and its black body spectrum lies in the nontrivial causal structure of the space-times containing black holes. The crucial point is the existence of the event horizons. The same takes place in the Rindler space-time. This space-time is obtained by transforming the two-dimensional Minkowski flat space-time from the "ordinary" coordinates (t, x) and metric $ds^2 = dt^2 - dx^2$ related to the set of inertial observers, to the so-called Rindler coordinates (η, ξ) $\left(t = \dfrac{1}{a} e^{a\xi} \sinh a\eta,\ x = \pm\dfrac{1}{a} e^{a\xi} \cosh a\eta,\ -\infty < \eta < \infty,\ -\infty < \xi < \infty\right)$ and metric $ds^2 = e^{2a\xi}(d\eta^2 - d\eta^2)$. Thus, the Rindler space-time is static and locally flat but differs from the two-dimensional Minkowski space-time globally, because it covers only one half of the latter and, in addition, possesses the event horizon at $t = \pm x$ ($\eta = \pm\infty$, $\xi = \mathrm{const}$). The Rindler observers at $\xi = \mathrm{const}$ are uniformly accelerated. The norm of the acceleration vector a^μ equals $\alpha = \sqrt{-a^\mu a_\mu} = ae^{-a\xi}$. Considering a quantum scalar field in the Rindler space-time, W.G. Unruh found [13], in fact, the finite temperature quantum field theory with the temperature

$$T_U = \frac{a}{2\pi}. \tag{3}$$

We see that this temperature is proportional to the acceleration of the Rindler observer sitting a $\xi = 0$ with $g_{00} = 1$. But, all of them are equivalent (we can always shift the spatial coordinate $\xi \to \xi - \xi_0$). The temperature is not an invariant, but it is a temporal component of a heat vector. This means that each observer measures the Unruh temperature when using its proper time τ ($ds = d\tau$). If the same observer uses the local clocks that show the local time t ($ds = \sqrt{g_{00}}\,dt$), the local temperature measured by him equals $T_{\mathrm{loc}} = \dfrac{T_U}{\sqrt{g_{00}}} = \dfrac{a}{2\,\pi}\, e^{-a\xi} = \dfrac{\alpha}{2\,\pi}$, which is proportional to the local acceleration α. The very fact that the uniformly accelerated observer (= detector) will detect the real particles in the vacuum, was known to people doing quantum electrodynamics long ago. It was understood as a change of a vacuum state due to the external forces that cause such an acceleration. The same happens in the space-time with event horizons. But that the spectrum is thermal appeared to be new and purely relativistic feature. We know from the university course of thermodynamics that the condition for thermal equilibrium in static space-times is $T_{\rm loc}\sqrt{g_{00}} = \text{const}$. Thus, all the Rindler observers are in thermal equilibrium with each other. Is the Rindler space-time unique in this sense? To answer, let us consider some general two-dimensional static space-time with a metric

$$ds^2 = e^{\nu}\, dt^2 - d\rho^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dq^2. \tag{4}$$

In the Rindler case $\rho = \frac{1}{a} e^{a\xi}$, $e^{\nu} = a^2\rho^2 = g_{00}$. The static observer undergoes a constant acceleration with the invariant $\alpha = \frac{1}{2}\left|\frac{d\nu}{d\rho}\right| = \frac{1}{2}\left|\frac{d\nu}{dq}\right| e^{-\lambda/2}$, and the (now local) Rindler parameter $a(\rho)$, which is called "the surface gravity $\varkappa$," is

$$\varkappa = \frac{1}{2}\left|\frac{d\nu}{dq}\right| e^{(\nu-\lambda)/2} = \frac{1}{2}\left|\frac{d\nu}{d\rho}\right| e^{\nu/2}. \tag{5}$$

The thermal equilibrium requires $\varkappa = \text{const}$, therefore, $g_{00} = C\rho^2$, and this proves that the Rindler space-time is the only one where static observers are in thermal equilibrium.

By the Einstein equivalence principle, we can extend all we learned studying Rindler space-times, to the static gravitational fields, especially to the spherically symmetric ones, because after fixing spherical angles θ and φ the latter become, in fact, the two-dimensional pseudo-surfaces. Of course, in general these surfaces are curved, the equivalence principle holds only locally, and the static observers will not be in thermal equilibrium with each other. Such a temperature is observer-dependent and cannot be considered as an intrinsic property of a given space-time. But for the black hole space-times, the position of the event horizon is absolute and does not depend on the observer. So, its temperature does serve an important characteristic of space-time itself. To know the temperature, we just need to compute the surface gravity value at the event horizon $\varkappa_H$. For the Schwarzschild black hole with the famous metric

$$ds^2 = F\, dt^2 - \frac{1}{F}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad F = 1 - \frac{2\,Gm}{r}, \tag{6}$$

where m is the black hole mass, and r is the radius of a sphere (in that sense that its area is $4\pi r^2$), the horizon is located at the radius $r_g = 2Gm$, and the surface gravity is

$$\varkappa_H = \frac{1}{2}\left|\frac{d\nu}{dr}\right| e^{(\nu-\lambda)/2} = \frac{1}{2}\, F'(r_H) = \left.\frac{Gm}{r^2}\right|_{r_g} = \frac{1}{4\,Gm}. \tag{7}$$

Therefore, the Hawking temperature is just the Unruh temperature at the event horizon measured by distant observers (at infinity). The same is true also for Kerr – Newman black holes. Note that outside the event horizon $r > r_g$ the Schwarzschild observers are not in thermal equilibrium with each other, and this is a thermodynamical explanation of the Hawking radiation and, thus, evaporation of black holes. It should be stressed that both the black hole temperature and entropy are global features because their very appearance is due to the existence of the event horizon.

Evaporating, black holes become smaller and smaller and will reach eventually a Planck size where the still unknown quantum gravity should play an important role. Since the radiation is quantized, the black hole mass have to be quantized as well. Of course, the relation is not direct because a black hole is not necessarily transformed into black hole again, but the new black hole will eventually be formed only due to radiation. Not only the rest masses and kinetic energy of particles, including the total angular momentum, may contribute to the black hole mass, but also Coulomb and magnetic energies of their electric and gauge charges and all kinds of other physical fields confined under the event horizon. But the common feature for all types of black holes is their entropy with its universal relation (2) to the horizon area. Thus, the black hole quantization means the quantization of its entropy. Moreover, the thermodynamical description is possible only if the jump in the temperature due to quantization of mass, charge and angular momentum during black hole evaporation is negligible compared to its absolute value, while the notion of the entropy as a measure of the information, hidden or ignored, is still valid. This latter feature gives rise to common believe that the black hole quasi-classical quantization can shed light on the structure of the future full quantum gravity, or, at least, will provide us with some selection rules in the attempts to construct such a theory. The quantization of a black hole as a whole was proposed long ago by J. Bekenstein [14]. The idea was based on the remarkable observation that the horizon area of non-extremal black holes behaves as a classical adiabatic invariant. The Bohr – Sommerfeld quantization rule then predicts the equidistant spectrum for the horizon area and thus, for the black hole entropy. The gedanken experiments show that, due to the quantum effects, the minimal increase in the horizon area in the processes of capturing a neutral or electrically charged particle approximately equals $\Delta A_{\min} \approx 4\,\ell_{p_\ell}^2$. This suggests for the black hole entropy

$$S_{BH} = \gamma_0\, N, \quad N = 1, 2, \ldots, \tag{8}$$

where γ_0 is of order of unity. In their famous work on the black hole spectroscopy, J.D. Bekenstein and V.F. Mukhanov [15] related the black hole entropy to the number g_n of microstates that corresponds to the particular external macrostate through the well-known formula in statistical physics $g_n = \exp[S_{BH}(n)]$; i.e., g_n is the degeneracy of the n-th area eigenvalue. Since g_n should be integer, they deduced that

$$\gamma_0 = \log k, \quad k = 2, 3, \ldots \tag{9}$$

In the spirit of the information theory and the famous claim by J.A. Wheeler "It from Bit," the value of $\log 2$ seems most suitable one.

The logarithmic behavior of the spacing coefficient γ_0 comes also from the Loop Quantum Gravity. It was shown in [16, 17] that the entropy of the Schwarzschild black hole is proportional to the horizon area as well as a numerical constant called the Barbero – Immirzi parameter. To fit the Bekenstein – Hawking relation (2) and the possible value for γ_0 (9) this parameter should equal $\log 2/(\pi\sqrt{3})$ if the fundamental group in LQG is $SU(2)$, and $\log 3/(2\pi\sqrt{2})$ if it is $SU(3)$. The choice of the value for γ_0 leads to minimal possible change in the black hole mass. S. Hod [18], using Bohr's correspondence principle, deduced that γ_0 should be proportional to $\log 3$ because he noticed that

$$Gm\,\mathrm{Re}\,w = 0.0437123 = \frac{\log 3}{8\pi}. \tag{10}$$

The value of γ_0 as well as that of Barbero – Immirzi parameter and, thus, the choice of the fundamental group in LQG must be universal. Therefore, it is not surprising that people tried to find some analytical methods for evaluating the quasi-normal frequencies for different types of black holes. By using rather sophisticated tools from the general theory of ordinary differential equations, L. Molt and A. Neitzke showed [19, 20] that for the scalar and tensor perturbations around Schwarzschild black holes the value $\log 3$ is exact. For more general types of black holes, the corresponding calculations were fulfilled in [21]. It appeared that the simple value $\log 3$ for the spacing coefficient γ_0 is by no means universal, but exceptional. That is why we use the expression "the mystery of $\log 3$."

Below, we construct a model which is not really a black hole, but possesses its main features. It has an event horizon – but local, the temperature – but local. Then, we develop the local thermodynamics for such a model and show how the mystery of $\log 3$ can be solved. There is a hope that our model will be helpful in understanding the underlining physics of many other interesting features of quasi-classical black holes.

2. The "Standard model"

2.1. Quantum shells. We start the construction of our model with a brief description of a particular model of quantum Schwarzschild black hole. Namely, this is a theory of quantized spherically symmetric self-gravitating thin dust shells [22, 23] – the simplest generalization of a point particle. In this case, there is only one dynamical degree of freedom, the shell radius (real gravitons are absent due to the spherical symmetry = Birkhoff theorem), and the Wheeler – DeWitt equation is reduced to the stationary one-dimensional Schroedinger-like equation in finite differences. Most important is the fact that the model is self-consistent, it takes into account the back reaction of the gravitating source (thin shell) on the geodesically complete Schwarzschild manifold which has a nontrivial causal structure. The geodesically complete Schwarzschild space-time has a geometry of non-transversable wormhole (it is also called an eternal black hole). There are two asymptotically flat regions with spatial infinities connected by the Einstein – Rosen bridge (the throat). Two sides of the bridge are causally disconnected and separated by (past and future) event horizons. Inside the shell we have some part of Schwarzschild metric with the mass parameter m_{in}, while outside the shell, the Schwarzschild mass is m_{out}.

In quantum mechanics, there are no trajectories, and the shell wave function "feels" the existence of the event horizons and both infinities. The result is the necessity of imposing an additional boundary condition and the appearance of two quantum numbers for two quantities describing the quantum states (for fixed m_{in}) – the bare mass ΔM of the shell (the sum of masses of the constituents) and its total mass (energy) $\Delta m = m_{\text{out}} - m_{\text{in}}$ which includes the gravitational mass defect. The discrete mass spectrum for bound sates looks as follows (n and p are integers):

$$\frac{2\,(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} = \frac{2\,m_{p_\ell}^2}{\Delta m + 2m_{\text{in}}}\,n,$$

$$M^2 - (\Delta m)^2 = 2\,(1+2p)\,m_{p_\ell}^2. \tag{11}$$

For given bare mass M, the change of a quantum state causes the change in the mass inside the shell m_{in} and in the total mass of the system m_{out}. Therefore, during the gravitational collapse the total mass decreases, while the inner mass increases. When could such a process be stopped? The natural limit is the crossing of the Einstein – Rosen bridge, since such a transition requires (at least in a quasi-classical regime) insertion of infinitely large volume, with, of course, zero probability. Computer simulations show that the process of quantum collapse for our shells stops when the principal quantum number becomes zero, $n = 0$.

The point $n = 0$ in our spectrum is very special. In this case the shell does not "feel" not only the outer region (what is natural for the spherical configuration), but it does not know anything about what is going on inside. It "feels" only itself. Such a situation reminds the "no hair" property of a classical black hole. Finally, when all the shells (both the primary one and newly born) are in the corresponding states $n_i = 0$, the whole system does not "remember" its own history. Then it is this "no-memory" state that can be called "the quantum black hole." Note that the total masses of all the shells obey the relation $\Delta\, m_i = \dfrac{1}{\sqrt{2}}\, M_i$.

2.2. Classical analog of quantum Schwarzschild black hole. The final state of quantum gravitational collapse can be viewed as some stationary matter distribution. Therefore, we may hope that for massive enough quantum black hole such a distribution is described approximately by a classical static spherically symmetric perfect fluid with energy density ε and (effective) pressure p obeying classical Einstein equations. This is what we call a classical analog of a quantum black hole. Of course, such a distribution has to be very specific. To study its main features, let us consider the situation in more details.

Any static spherically symmetric metric can be written in the form

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{12}$$

Here r is the radius of a sphere with the area $A = 4\pi r^2$, $\nu = \nu(r)$, $\lambda = \lambda(r)$. There are only three (static spherically symmetric) Einstein equations. The constraint equation can be written in the integral form. For this, let us integrate the first of Eqs. (11):

$$e^{-\lambda} = 1 - \frac{2\,Gm(r)}{r}, \tag{13}$$

where

$$m(r) = 4\pi \int_0^r \varepsilon\, \tilde{r}^2\, d\tilde{r} \tag{14}$$

is the mass function that should be identified with $m_{\rm in}$. Now, the "no-memory" principle is readily formulated as the requirement that $m(r) = ar$, i.e.,

$$e^{-\lambda} = 1 - 2Ga = \text{const}, \quad \varepsilon = \frac{a}{4\pi r^2}. \tag{15}$$

We can also introduce a bare mass function $M(r)$ (the mass of the system inside a sphere of radius r without gravitational mass defect):

$$M(r) = \int \varepsilon\, dV = 4\pi \int_0^r \varepsilon\, e^{\lambda/2}\, \tilde{r}^2\, d\tilde{r} = \frac{ar}{\sqrt{1-2Ga}}. \tag{16}$$

The remaining two equations can now be solved for $p(r)$ and $e^{\nu}(r)$. The general solution is rather complex, but the correct non-relativistic limit for the pressure $p(r)$ (we are to reproduce the famous equation for hydrostatic equilibrium) is given by only the following one-parameter family:

$$p(r) = \frac{b}{4\pi r^2}, \quad b = \frac{1}{G}\left(1 - 3Ga - \sqrt{1-2Ga}\,\sqrt{1-4Ga}\right). \tag{17}$$

We see that the solution exists only for $a \le \frac{1}{4G}$, then $b \le a$. The physical meaning of these inequalities is that the speed of sound cannot exceed the speed of light, $v^2_{\rm sound} = \frac{b}{a} \le 1 = c^2$, the equality being reached just for $a = b = \frac{1}{4G}$. Finally, for the temporal metric coefficient $g_{00} = e^{\nu}$ we get

$$e^{\nu} = C_0^2\, r^{4b/(a+b)} = C_0^2\, r^{2G(a+b)/(1-2Ga)}.$$

Thus, demanding the "no-memory" feature and the existence of the correct non-relativistic limit, we obtained the two-parameter family of static solutions. But, we need a one-parameter family, so we have to continue our search.

Evidently, the point $r = 0$ is singular both for matter distribution and g_{00} metric coefficient. To examine what kind of singularity we are dealing with, one should calculate the Riemann curvature tensor. It appears that for $b < a$ this tensor is, indeed, divergent at $r = 0$. But, if $a = b = \dfrac{1}{4G}$, we are witnessing a miracle, the (before) divergent components become zero. Thus, demanding, in addition to the previous two very natural requirements, the third one (also natural), namely, the absence of the real (curvature) singularity at $r = 0$, we arrive at the following one-parameter family of solutions to the Einstein equations (11)

$$e^{\nu} = C_0^2 r^2, \quad e^{\lambda} = 2, \quad \varepsilon = p = \frac{1}{16\pi G r^2}. \tag{18}$$

So, the equation of state of our perfect fluid is the stiffest possible one. The constant of integration C_0 can be determined by matching the interior and exterior metrics at some boundary value of radius, $r = r_0$. Let us suppose that for $r > r_0$ the space-time is empty, so, the interior should be matched to the Schwarzschild metric with the mass parameter m. Of course, to compensate the jump in the pressure $\Delta p \ (= p(r_0) = p_0)$, we must include in our model a surface tension Σ; so, actually, we are dealing with some sort of liquid. It is easy to check that

$$C_0^2 = \frac{1}{2r_0^2}, \quad \Delta p = \frac{2\,\Sigma}{\sqrt{2} r_0}, \quad e^{\nu} = \frac{1}{2}\left(\frac{r}{r_0}\right)^2,$$

$$p_0 = \varepsilon_0 = \frac{1}{16\pi G r_0^2}; \quad m = m_0 = \frac{r_0}{4G}.$$

Note that the bare mass $M = \sqrt{2} m$, the relation is exactly the same as for the shell "no-memory" state and $r_0 = 4Gm_0$, so, the size of our analog model is twice as that for a classical black hole of the same mass.

The special point in our solution $r = 0$ is not a trivial coordinate singularity, like in a three-dimensional spherically symmetric space, because $ds^2\,(r = 0) = 0$. This looks like an event horizon. Indeed, the two-dimensional $(t - r)$-part of our metric describes a locally flat manifold. Since the static observers at $r = \text{const}$ are, in fact, uniformly accelerated, this is a Rindler space-time with the event horizon at $r = 0$. The corresponding Rindler parameter which in more general case is called the "surface gravity," equals

$$\varkappa = \frac{1}{2}\left|\frac{d\nu}{dr}\right| e^{(\nu-\lambda)/2} = \frac{C_0}{\sqrt{2}} = \frac{1}{2r_0}. \tag{19}$$

Therefore, the Unruh temperature in our model is $T_U = \dfrac{1}{4\pi r_0} = \dfrac{1}{16\pi G m}$, what is twice less than the Hawking temperature for the Schwarzschild black hole,

$$T_H = \frac{1}{8\pi G m} = 2T_U. \tag{20}$$

Let us resume what we have got by now. We constructed a purely classical model that possesses some features of (semi)classical black holes: event horizon and temperature, but instead of being global, they are local. Indeed, by definition, the surface $r = 0$ cannot be crossed; thus, the event horizon in our model becomes local. The temperature is also local, $T_{\text{loc}} = T_U/\sqrt{g_{00}} = 1/2\sqrt{2}\pi r$, and does not depend on the boundary value r_0. And, one more important feature: if one removes some outer layer, nothing would be changed inside. This is a reflection of the fact that all parts of our matter distribution are in thermal equilibrium.

Quantum nature of radiation and the fact that the black hole entropy has a discrete equidistant spectrum suggest that our distribution consists, actually, of some number of Quasi-particles, “gravitational phonons.” Thus, having at hand local intensive parameters: effective pressure $p(r)$, temperature $T_{\rm loc}(r)$, chemical potential $\mu(r)$, and extensive parameters: bare mass M, volume V, entropy S and “particle” number N, we are now ready to construct the local thermodynamics.

2.3. Thermodynamics. The first law of thermodynamics reads

$$dM = \varepsilon\, dV = T_{\rm loc}\, dS - p\, dV + \mu\, dN. \tag{21}$$

Dividing the above expression by the volume element dV we get the first law in its local form

$$\varepsilon(r) = T_{\rm loc}(r)\, s(r) - p(r) + \mu(r)\, n(r), \tag{22}$$

where s and n are the entropy and particle densities, respectively. In our model $\varepsilon = p$, but what about s? The local observer cannot calculate it without knowing the corresponding microscopic structure, but he can ask his global counterpart who is educated enough (reads proper books) and knows that the total entropy of the black hole is $S = \dfrac{1}{4G} A_{\rm hor}$, what for the Schwarzschild black hole gives $(A_{\rm hor} = 4\pi r_g^2)$ $S = \dfrac{\pi}{G} r_g^2 = \dfrac{\pi\, r_0^2}{4G}$. Having this information, our local observer can deduce that

$$s(r) = \frac{1}{8\sqrt{2}\, Gr}, \quad T_{\rm loc}(r)\, s(r) = \frac{1}{32\pi G r^2}. \tag{23}$$

Remembering now that $\varepsilon = \dfrac{1}{16\pi G r^2}$, we obtain

$$T_{\rm loc}(r)\, s(r) = \frac{1}{2}\,\varepsilon, \quad \mu(r)\, n(r) = \frac{3}{2}\,\varepsilon.$$

We will need also the expression for the free energy F:

$$F = \int f\, dV, \quad f = \varepsilon - T_{\rm loc}\, s = \frac{1}{2}\,\varepsilon. \tag{24}$$

It is known that the thermal equilibrium conditions for the systems in static gravitational field are

$$T\sqrt{g_{00}} = \text{const}, \quad \mu\sqrt{g_{00}} = \text{const}. \tag{25}$$

The constants on the right-hand sides are universal for our model – they do not depend on the boundary value r_0. Therefore, their ratio is also a universal constant. Thus, we have

$$\frac{\mu}{T} = 3\,\frac{s}{n} = 3\,\frac{S}{N} = 3\gamma_0. \tag{26}$$

Hence, the entropy is naturally quantized:

$$S = \gamma_0\, N, \quad N = 1, 2, \ldots \tag{27}$$

2.4. Solving the mystery of log 3. In order to calculate the spacing coefficient γ_0 we have to make some assumption about the microscopic structure of our model. We assume that the interior matter distribution consists of N black hole phonons with the equidistant spectrum of excitations

$$\varepsilon_n = \omega\, n, \quad n = 1, 2, \ldots \tag{28}$$

In this case, the partition function for the whole system is the product of those ones for each phonon, and

$$Z_{\text{tot}} = (Z_1)^N, \quad Z_1 = \sum_n e^{-\varepsilon_n/T} = \sum_n \left(e^{-\omega/T}\right)^n = \frac{e^{-\omega/T}}{1 - e^{-\omega/T}}. \tag{29}$$

It is natural to suppose that ω is just the black hole resonance frequency and its existence follows from the properties of quasi-normal modes (as was already explained earlier). Of course, ω is a temporal component of a four-vector, the same is the temperature T, so their ratio does not depend on the choice of the clocks by local static observers. We accept that the observers are using their proper time, so T is just the Unruh temperature T_U which is constant in the whole interior. The partition function is an invariant, and we can calculate it in another way, using thermodynamical relations. Indeed, we can consider some small volume element dV and the corresponding partition function Z_{small}. Then, using the well-known formula for the free energy $F = -T \log Z$, and writing it for the volume element

$$dF = f\, dV = -T_{\text{loc}} \log Z_{\text{small}}, \tag{30}$$

where, as before, we use the local intrinsic quantities in thermodynamical relations. From this we have

$$\int \frac{f}{T_{\text{loc}}}\, dV = -\sum \log Z_{\text{small}} = -\log Z_{\text{tot}}. \tag{31}$$

The left-hand side is

$$\int \frac{f}{T_{\text{loc}}}\, dV = \frac{1}{2} \int \frac{\varepsilon}{T_{\text{loc}}} = \frac{\pi\, r_0^2}{4G} = \frac{\pi\, r_g^2}{G} = S. \tag{32}$$

Here r_g is the Schwarzschild radius, and S is the total black hole entropy. Eventually, we obtain the important relation

$$e^{-S} = Z_{\text{tot}} = (Z_1)^N, \tag{33}$$

from which it follows that

$$\frac{e^{-\omega/T}}{1 - e^{-\omega/T}} = e^{-S/N} = e^{-\gamma_0}, \quad e^{\gamma_0} = e^{\omega/T} - 1. \tag{34}$$

To go further, let us consider the irreversible process of converting the mass (energy) of the system into radiation from a thermodynamical point of view. In our model such a process takes place just at the boundary $r = r_0$, and the thin shell with zero surface energy density and surface tension Σ serves as a converter supplying the radiation with extra energy and extra entropy, this resembles the "brick wall" model. The nature of this radiation is purely quantum because our system is not radiating classically. The jump in the Unruh temperature of the inner and outer near-boundary static observers is compensated exactly by the gravitational influence of the surface tension. One can imagine that the near-boundary layer of thickness Δr_0 is converting into radiation, thus decreasing the boundary of the inner region to $(r_0 - \Delta r_0)$. Its energy is $\Delta M = \varepsilon \Delta V$. To this we should add the energy released from the work done by the surface tension due to its shift, which is equal exactly to $\sum d(4\pi r_0^2) = p d\Delta V = \varepsilon \Delta V = \Delta M$. Therefore, both the energy and the temperature in the converter becomes two times higher than that for any inner layer of the same thickness. And this double energy is gained by radiating quanta. Clearly, they have double frequency and exhibit double temperature, so

$$\frac{\operatorname{Re} w}{T_H} = \frac{\omega}{T_U} = \log 3, \tag{35}$$

as follows from the spectrum of quasi-normal modes for the Schwarzschild black holes. Substituting this into Eq. (34) and remembering that

$$3 - 1 = 2, \tag{36}$$

we obtain

$$\gamma_0 = \log 2. \tag{37}$$

Since the radiated energy is thermalized, the interpretation of dm as equal to $\mathrm{Re}\, w$ is an improper procedure. This resolves the "$\log 3$-paradox."

3. Beyond the "Standard model"

The model proposed above is very stringent. And the question arises: which of the imposed conditions could be weakened? Let us remember the steps towards the final results. First, we demanded the "no memory" condition to be fulfilled. This was necessary to ensure the black hole mimicry. Second, we assumed the perfect fluid energy-momentum tensor. Then, the requirement for the absence of a curvature singularity at zero radius has led us both to the appearance of the temperature and to the unique (stiffest possible) equation of state. Surely, the thermal equilibrium is the crucial feature, but how about the isotropy in the fluid pressure?

To make this point clearer, let us consider the general form of static spherically symmetric metric with static observers in mutual thermal equilibrium. As we already know, the space-time in such a case should be a direct product of Rindler (locally flat) manifold and 2-dimensional sphere of radius R:

$$ds^2 = a^2\rho^2\, dt^2 - d\rho^2 - R^2(\rho)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{38}$$

where a is the acceleration parameter, and $R(\rho)$ is the only unknown function of the radial coordinate ρ. The Einstein equations read as follows:

$$\begin{aligned} &-\frac{2R''}{R} + \frac{1 - R'^2}{R^2} = 8\pi G\varepsilon, \\ &-2\frac{R'}{\rho\, R} + \frac{1 - R'^2}{R^2} = -8\pi G p_r, \\ &-\frac{R''}{R} - \frac{R'}{\rho\, R} = -8\pi G p_t. \end{aligned} \tag{39}$$

Here "prime" denotes ordinary derivatives and we assume that, in general, the radial pressure p_r is not equal to the tangential pressure p_t. With the "no memory" condition $R' = \alpha = \text{const}$, the above equations become algebraic; besides, in this case $\varepsilon + p_r = 2p_t$ and for isotropic pressure $p_r = p_t$ we recover the previous result. But, let us remember that the relation between the bare and total masses $M = \sqrt{2}m$ in our model appeared the same as that of the quantized thin dust shells in the "no memory" states. And this does not point to the fact that our classical analog consists solely of massive constituents. But in reality, classical black holes may contain some radiation (i.e., massless particles) as well. Consider now the extreme situation when the analog model distribution is represented by massless particles only. Then, $\varepsilon = p_r + 2p_t$ and, hence, $p_r = 0$, $\varepsilon = 2p_t$ $\alpha = 1/\sqrt{3}$. Such a strange equation of state means that we are dealing not with a condensed matter but rather with a set of thin shells of small (vanishing) energies that consist of massless particles orbiting along the spheres of constant radii in all possible directions [25]. But such a distribution is unstable, because the orbits coincide with the last circular ones in the outer Schwarzschild metric. In the intermediate case, there is a mixture, and these orbits become stable. Moreover, if one assumes that these two systems are non-interactive (except gravitationally), what seems quite natural in the spirit of our "no memory" condition, then it is not difficult to show, using separate continuity equations, that $R' = \mathrm{const}$ and the perfect fluid part of the mixture has the stiffest possible equation of state.

Such a generalized model possesses plausible features. First, the value for R' is no more unique, instead, $1/3 < \alpha^2 \le 1/2$. Second, these orbiting massless particles can be understood as remnants of radiated quasi-normal modes and, at the same time, as the origin of the equidistant "phonon" spectrum in the perfect fluid. Third, the "Hawking evaporation" of our analog model can now be considered as the induced radiation tunneling trough the potential barrier caused by the surface tension at the boundary.

It is not yet clear how to make use of the thermodynamical relations in this rather complex system and... but the work is in progress.

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Резюме

В.А. Березин. Замечания о классических аналогах квантовых чёрных дыр.

Построена модель, в которой основные глобальные свойства классических и квазиклассических черных дыр становятся локальными (горизонт событий, отсутствие «волос», температура и энтропия). Наша схема базируется на особенностях квантового коллапса, обнаруженных при изучении некоторых конкретных моделей квантовых черных дыр. Однако наша модель является чисто классической, что позволяет использовать самосогласованным образом уравнения Эйнштейна и классическую (локальную) термодинамику и таким образом объяснить «проблему $\log 3$».

Ключевые слова: классические и квазиклассические чёрные дыры.

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